数理解析研究所講究録
1159 巻 2000 年 135-140

THE CONTINUATION OF HOLOMORPHIC SOLUTIONS TO CONVOLUTION EQUATIONS IN COMPLEX DOMAINS

Ryuichi ISHIMURA, Jun-ichi OKADA, Yasunori OKADA
(石村隆一, 岡田純一, 岡田靖則)

1 Introduction

First of all, The problem of analytic continuation of the solutions to a homogeneous linear partial differential equation with constant coefficients was considered by Kiselman [7]. He proved that the directions to whom every solution is analytically continued are determined by *its characteristic set.* (See also Zerner [12].) After that, under an additional hypothesis, Sébbar [11] extended the method of [7] to the case of local differential operators of infinite order with constant coefficients. Motivated by [11], Aoki [1] proved a local continuation theorem for the general differential operators of infinite order with variable coefficients, using his theory of exponential calculus for pseudo-differential operators. In the case of convolution equation with a hyperfunction kernel defined in tube domains invariants by any real translations, Ishimura and Y. Okada [2] proved that the directions to whom not every solution can be continued at once were contained to *the characteristic set* of the operator, by using the method developped by [7] and [11].

In this talk, we consider the homogeneous convolution equation $S * f = 0$ with an analytic functional S and study the analytic continuation of the solution f.

We refer to [5] for the details and the proof.

2 The characteristic set and the condition (S)

In this section, we shall introduce the characteristic set and the condition $(S)_{\zeta_0}$. For any open set $\omega \subset \mathbb{C}^n$, we denote by $\mathcal{O}(\omega)$ the space of holomorphic functions defined on ω. Let S be an analytic functional on $\mathbb{C}^n$ and we suppose that S is supported by a compact convex set $K \subset \mathbb{C}^n$. $\hat{S}$ denote its Fourier-Borel-transform

$$\hat{S}(\zeta) = < S, \exp(z \cdot \zeta) >_z, \tag{2.1}$$

which is an entire function of exponential type satisfying the following estimate (the theorem of Polyà-Ehrenpreis-Martineau). For every ε, we can take a constant $C_\varepsilon > 0$ such that

$$|\hat{S}(\zeta)| \leqq C_\varepsilon \exp\left(H_K(\zeta) + \varepsilon|\zeta|\right), \tag{2.2}$$

where $H_K(\zeta) = \sup_{z \in K} \mathrm{Re} < z, \zeta >$ is the supporting function of K.

For a set $A \subset \mathbb{C}^n$, we set $A^a = -A$. we define the convolution operator $S*$ by

$$(S * f)(z) = < S, f(z - w) >_w \qquad \text{for } f \in \mathcal{O}(\omega + K^a), \tag{2.3}$$

and consider the homogeneous convolution equation

$$S * f = 0. \tag{2.4}$$

We define the sphere at infinity

$$S_\infty^{2n-1} = (\mathbb{C}^n \setminus \{0\})/\mathbb{R}_+$$

and denote by $\zeta\infty$ the equivalent class of $\zeta \in \mathbb{C}^n \setminus \{0\}$. We consider the compactification with directions

$$\mathbb{D}^{2n} = \mathbb{C}^n \sqcup S_\infty^{2n-1}$$

of $\mathbb{C}^n$.

Let $f(\zeta)$ be an entire function of exponential type. In accordance with Lelong and Gruman [9], we define the growth indicator of f by

$$h_f(\zeta) = \limsup_{r \to \infty} \frac{\log |f(r\zeta)|}{r}, \tag{2.5}$$

and the regularized growth indicator of f by

$$h_f^*(\zeta) = \limsup_{\zeta' \to \zeta} h_f(\zeta). \tag{2.6}$$

As in [2], and generalizing to the present case, we define the characteristic set of $S*$:

Definition 2.1. We set

$$\begin{aligned}\mathrm{Char}_\infty(S*) = &\text{the complement of } \{\tau\infty \in S_\infty^{2n-1} \ ;\\ &\text{for every } \varepsilon > 0, \text{ there exist } N > 0 \text{ and } \delta > 0 \text{ such that}\\ &\text{for any } r > N \text{ and } \zeta \in \mathbb{C}^n \text{ satisfying } \left|\zeta - \frac{\tau}{|\tau|}\right| < \delta,\\ &\text{we have } |\hat{S}(r\zeta)| \geqq \exp(h^*_{\hat{S}}(\zeta) - \varepsilon)r\}\end{aligned}$$

and call it the characteristic set of the operator $S*$.

Now we recall the definition of the condition (S), originally due to T. Kawai [6] and was defined in a direction in [4].

Definition 2.2. We say that an entire function f of exponential type satisfies the condition (S) at direction $\zeta_0 \in \mathbb{C}^n \setminus \{0\}$, if it satisfies the following:

$$(S)_{\zeta_0} \qquad \begin{cases} \text{For every } \varepsilon > 0, \text{ there exisits } N > 0 \text{ such that} \\ \text{for any } r > N, \text{ we have } \zeta \in \mathbb{C}^n \text{ satisfying} \\ |\zeta - \zeta_0| < \varepsilon, |f(r\zeta)| \geqq \exp\left(h^*_f(\zeta_0) - \varepsilon\right) r. \end{cases}$$

Remark . This condition is equivalent to the condition of regular growth which is the classialc notion in the theory of entire functions (see [4]).

Remark . By (2.2) and (2.6), we have in general $h^*_{\hat{S}}(\zeta) \leqq H_K(\zeta)$. Hereafter we shall make assumption $h^*_{\hat{S}}(\zeta) \equiv H_K(\zeta)$. For open convex domains, this condition and the condition (S) are , in a sense, necessary and sufficient conditions for the solvability of inhomogeneous convolution equation $S * f = g$. See Krivosheev [8] for the more precise statement.

3 Main theorem and example

For the characteristic set $\mathrm{Char}_\infty(S*)$ and an open convex set $\omega \subset \mathbb{C}^n$, we set

$$\Omega = \text{the interior of } \left(\bigcap_{\zeta\infty \in \mathrm{Char}_\infty(S*)^a} \{z \in \mathbb{C}^n \ ; \ \mathrm{Re} < z, \zeta > \leqq H_\omega(\zeta)\} \right). \quad (3.1)$$

Our main theorem is the following:

Theorem 3.1. *Let $K \subset \mathbb{C}^n$ be a compact convex set and S an analytic functional supported by K. We suppose that S satisfies the condition $(S)_{\zeta_0}$ in any directions in $\mathbb{C}^n$ and $h^*_{\hat{S}}(\zeta) \equiv H_K(\zeta)$. For an open convex set $\omega \subset \mathbb{C}^n$, we define the open set Ω by (3.1). Then every holomorphic solution f to $S * f = 0$ defined on $\omega + K^a$ extends analytically to $\Omega + K^a$.*

Example . Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$ be a finite set in $\mathbb{C}^n$, K its convex-hull and $p_j(\zeta)$ an entire function of minimal type for $1 \leqq j \leqq l$. For the analytic functional S, we suppose its Fourier-Borel transform $\hat{S} = \sum_{j=1}^{l} p_j(\zeta) \exp < \zeta, \lambda >$. Then S is supported by K and by Ronkin [10] and by [4], we also know $h^*_{\hat{S}}(\zeta) \equiv H_K(\zeta)$ and that $\hat{S}$ satisfies the condition $(S)_{\zeta_0}$ in any directions in $\mathbb{C}^n$. Therefore this analytic functional S satisfies all hypothesis of the theorem above.

In particular, in case where p_j's are elliptic, that is to say, its characteristic set is empty, we can prove that the characteristic set $\mathrm{Char}_\infty(S*)$ coincides with the following:

$$\left\{\zeta\infty \in S^{2n-1}_\infty \; ; \; \#\{j \; ; \; \mathrm{Re} < \zeta, \lambda_j >= H_K(\zeta)\} \geqq 2\right\}.$$

See [3] for more detailed results. In the case of $n = 1$, $l = 4$ and $K =$ the convex-hull of Λ, the figures are the following:

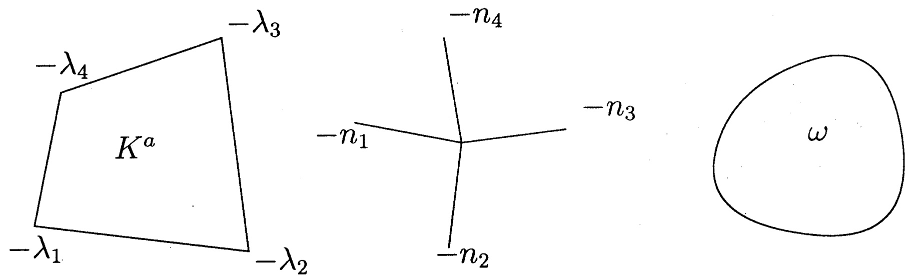

Figure 1: K^a, $\mathrm{Char}(S*)^a$ and ω

In this case, we remark

$$\mathrm{Char}_\infty(S*) = \text{the exterior normal directions } \{n_1\infty, n_2\infty, n_3\infty, n_4\infty\}.$$

In Figure 2, every solution $f \in \mathcal{O}(\omega + K^a)$ of $S * f = 0$ can be analytically continued to four corners.

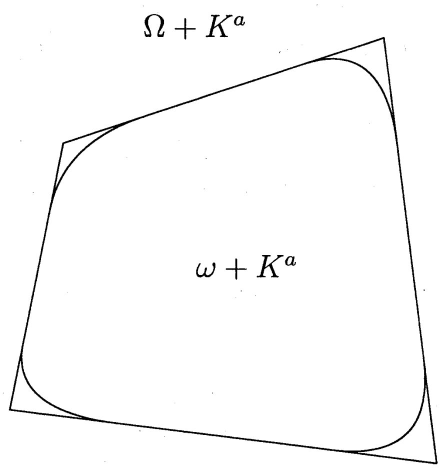

Figure 2: $\omega + K^a$ and $\Omega + K^a$

References

[1] T. Aoki, *Existence and continuation of holomorphic solutions of differential equations of infinite order*, Adv. in Math., **72**(1988), 261 – 283.

[2] R. Ishimura and Y. Okada, *The existence and the continuation of holomorphic solutions for convolution equations in tube domais*, Bull. Soc. math. France, **122**(1994), 413 – 433.

[3] R. Ishimura and Y. Okada, *Examples of convolution operators with described characteristics*, in preparation.

[4] R. Ishimura and J. Okada, *Sur la condition* (S) *de Kawai et la propriété de croissance régulière d'une fonction sous-harmonique et d'une fonction entière*, Kyushu J. Math., **48**(1994), 257 – 263.

[5] R. Ishimura, J. Okada, and Y. Okada, *The continuation of holomorphic solutions to convolution equations in complex domains*, preprint.

[6] T. Kawai, *On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **17**(1970), 467 – 517.

[7] C. O. Kiselman, *Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants*, Bull. Soc. Math. France, **97**, 1969, p. 329 – 356.

[8] A. S. Krivosheev, *A criterion for the solvability of nonhomogeneous convolution equations in convex domains of* $\mathbf{C}^n$, Math. USSR Izv., **36**(1991), 497 – 517.

[9] P. Lelong and L. Gruman, *Entire functions of several complex variables*, Grung. Math. Wiss., Berlin, Hidelberg, New York, Springer vol.282, 1986.

[10] L. I. Ronkin, *Functions of completely regular growth*, MIA, Kluwer, 1992.

[11] A. Sébbar, *Prolongement des solutions holomorphes de certains opérateurs différentiels d'ordre infini à coefficients constants*, Séminaire Lelong-Skoda, LNM822, Springer, Berlin (1980),199–220.

[12] M. Zerner, *Domaines d'holomorphie des fonctions vérifiant une équation aux dérivées partielles*, C. R. Acad. Sc., Paris, **272**(1971), 1646 – 1648.

Ryuichi ISHIMURA
Department of Mathematics and Informatics,
Faculty of Sciences, Chiba University
Yayoi-cho, Inage-ku, Chiba 263-8522, Japan
E-mail address: ishimura@math.s.chiba-u.ac.jp

Jun-ichi OKADA
Institute of Natural Sciences,
Yayoi-cho, Inage-ku, Chiba 263-8522, Japan
E-mail address: mokada@math.s.chiba-u.ac.jp

Yasunori OKADA
Department of Mathematics and Informatics,
Faculty of Sciences, Chiba University
Yayoi-cho, Inage-ku, Chiba 263-8522, Japan
E-mail address: okada@math.s.chiba-u.ac.jp